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(14) $\therefore \rho = a \sin\left((\tfrac{1}{2}\pi - \theta)\frac{p}{p+2}\right)\cdot$ Let $\tfrac{1}{2}\pi - \theta = \theta'$

(15) $$\rho = a \sin\frac{p}{p+2}.\theta' .$$

In this θ' being the complement of θ is to be measured from the axis of y.

Since p is essentially positive, the coefficient $p \div (p + 2)$ must be less be less than unity, and hence n must be a proper fraction.

We have $p = \frac{2n}{1 - n}$ and $a = \frac{2r_1(p+1)}{p} = \frac{r_1(n+1)}{n}$

$\therefore$ if $n = 1$, $p = \infty$, $\rho = a \sin\theta' = a\cos\theta$, and $a = 2r_1$,
" $n = \frac{1}{2}$, $p = 2$, $\rho = a\sin\frac{1}{2}\theta'$, " $a = 3r_1$,
" $n = \frac{1}{3}$, $p = 1$, $\rho = a\sin\frac{1}{3}\theta'$, " $a = 4r_1$,
&c., thus our demonstration is complete.

DETERMINATION OF ROOT OF N^{th} DEGREE.

BY DR. H. EGGERS, MILWAUKE, WISCONSIN.

1. I propose to give here an elementary method of extracting the root of any degree of a given number by an elementary geometrical process. This method is based on the following well known theorem of Ceva:

"A triangle ABC and an arbitrary point P in its plane are given. If we draw from the three vertices A, B, C, of the triangle three transversal lines through the point P, then on each side of the triangle two segments are formed: a and b; a_1 and b_1; a_2 and b_2; which fulfil the relation:

$$\frac{a}{b}\cdot\frac{a_1}{b_1}\cdot\frac{a_2}{b_2} = -1"$$

This theorem solves immediately the problem:

" To constauct two lines, the ratio of which is the product of two given ratios"; which solution requires no explanation.

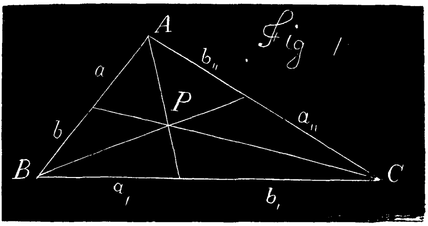


Fig. 1

If we make the sides of the triangle equal, we can *square a given ratio*. If $a \div b$ is the given ratio, we construct an isosceles triangle with two sides equal to $a + b$. If now we draw two lines from the vertices A and C to the dividing points D and E, where AD = BE = a, and DB=EC=b, of the opposite sides, we find the point of intersection P, and the line connecting B with P marks on AC the point F, which solves the problem.

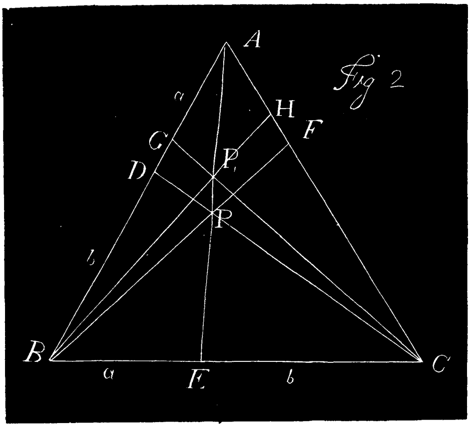
For according to Ceva's theorem we have,

$$\frac{a}{b}\cdot\frac{a}{b}=\left(\frac{a}{b}\right)^2=\frac{b_2}{a_2}.$$

In order to form the third power of the ratio $a \div b$, we have to construct an equilateral triangle with the side $(a+b)$. After having constructed according to above the square $(a \div b)^2 = \mathrm{AF} \div \mathrm{CF}$, we make the segment AG = AF and draw the line GC, which meets AE in P_1. The line BP_1 marks on HC the point H, which point divides AC in such a manner, that

$$\frac{\mathrm{AH}}{\mathrm{CH}}=\frac{\mathrm{AG}}{\mathrm{BG}}\cdot\frac{\mathrm{BE}}{\mathrm{CE}}=\left(\frac{a}{b}\right)^2\cdot\frac{a}{b}=\left(\frac{a}{b}\right)^3.$$

Proceeding in the same manner with AH as with AF before, we arrive at the construction of $(a \div b)^4$ After this we can construct the 5th power of $a \div b$ and so on. To form the *nth* power of $a \div b$, this same process has to be repeated $n - 2$ times.

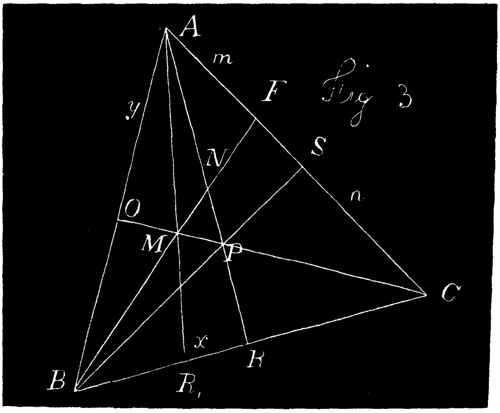


2. Let us now proceed to the inverse operation, *to find the root of a given degree of a given ratio* $m \div n$.

First let us extract the square root. We construct on the side $\mathrm{AC} = m + n = \mathrm{AF} + \mathrm{FC}$ an equilateral (or only an isoscele) triangle ABC, fig. 3, divide AB at an arbitrary point O, make BR = AO, draw the transversals AR and CO and through the point of intersection P, of the two latter we draw the line BP. This line BP meets AC in some point S. If S would coincide with F then the ratio

$$\frac{\mathrm{AO}}{\mathrm{BO}}=\frac{x}{y}$$

would be the exact square root of $m \div n$. But this will in general not be the case. To gain an idea of the error committed, we draw BF. This line meets AR and CO respectively in M and N. The triangle PMN we will call the triangle of error. It is evident that the size of this triangle depends on the error of the assumed root. The area of the triangle PMN will vanish together with the error and will increase and diminish with the error of the root. In order to find the correction of the root we draw AM which meets BC in R_1. The distance RR_1 may be called the *segment of* *error*, or briefly,the *error*, and may be designated by e, tht is, $RR_1 = e$.



In order to find a more approximate value of the root *we halve* RR_1 in X; then is

$$\frac{BX}{CX}$$

a more approximate value of the square root than the initial value BR ÷ CR.

We now try this new value by making AY = BX and gain another point S_1 which will be nearer to the point F, than the point S was. Supposing the point S_1 to have a measurable distance from F, we determine as above by a new point X_1 the new error e_1, which furnishes a still more approximate value $BX_1 \div CX_1$; this process has to be repeated till the point S_k coincides in proxi with F. The last found ratio $X_k \div Y_k$ is the square root of AF ÷ CF as near as construction can furnish it.

That this process is always convergent toward the required root needs to be demonstrated, but it would on this occasion require too much space and therefore may be left a problem for the reader. It may be stated here, that if the first trial value is any approximate value of the root, the next corrected value is so near the exact root, that the theoretical error of the algorithm is smaller than the error in drawing, so that a continued operation would be entirely useless.

3. From the above method we can immediately deduce the algorithm of finding the cube root of a given ratio. Let $m \div n$ be again the given ratio, Then we construct an equilateral triangle on the side $m + n = AC = AF + FC$. Now take an arbitrary approximate value $x \div y$ and construct the third power

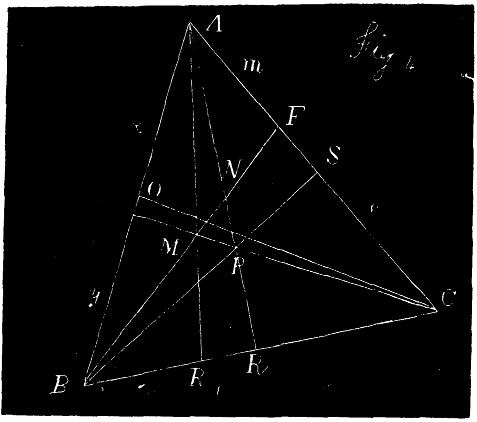
$$\left(\frac{x}{\ }\right)^3 = \frac{AS}{CS}.$$

The triangle of error is PMN. Now join AM and mark the point of intersection R_1 on BC. Then is again $RR_1 = e$ the segment of error. From this, cut off $R_1X = \frac{1}{3}R_1R$, then is

$$\frac{BX_1}{CX_1} = \frac{x_2}{y_2}$$

the correct root. If we now try this new value $x_1 \div y_1$ and should find that the new point S_1 on AC, corresponding to the point S of the first operation be not yet sufficiently near to F we have to determine a new segment of error XR_2, and by trisecting XR_2 would find a new correcting point X_1, so that the third approximate value of the cube root would be

$$\frac{BX_1}{CX_1} = \frac{x_2}{y_2}.$$



Continued repetition of this construction furnshes a value for the cube root as approximate as required. It may be observed again, that one single construction will be sufficient; for then the error in drawing will exceed the theoretical error of the algorithm, so that in practice this construction equals or surpasses any rigorous construction by means of higher curves, even if these latter could be easily drawn.

Remark. This construction of the cube root can be applied to the renowned Delian problem: To find the side of a cube which shall be the double of a given cube. For if a be the side of a given cube, then

$$x = a\sqrt[3]{2}$$

is the side of the required cube, and we have only to extract graphically the cube root of 2, etc.

Graphical extraction of the n^{th} root of a given ratio $\frac{m}{n}$.

4. To proceed at once to the general problem to extract the root of the n^{th} degree of a given ratio $m \div n$, we construct again an equilateral triangle with the side $m + n$, or with the sum of any convenient proportionals of m and n. Then take an approximate value $x \div y$ and form of this the n^{th} power, determine in like manner as above the segment of error, e; take the n^{th} part of e, add this part to the numerator x and subtract it from the denominator y, or the reverse, as the case may be, so that the next trial value, or the corrected root, is

$$\frac{x_1}{y_1} = \frac{x + (e \div n)}{y - (e \div n)};$$

the next corrected value would be

$$\frac{x_2}{y_2} = \frac{x_1 + (e_1 \div n)}{y_1 - (e_1 \div n)} \text{ \&c.}$$

The acceleration of this algorithm toward the required root is of quadratic order.

5. This geometrical algorithm may easily be translated into an algebraical algorithm and thus furnish an elementary arithmetical method of extracting roots by a method of very rapid convergence. The general algorithm thus obtained is as follows:

$$\frac{x_1}{y_1} = \frac{(n-1)x^n + y^n z + nzxy^{n-1}}{nx^{n-1}y + x^n + (n-1)y^n z}, \tag{1}$$

where n denotes degree of root, z the given number, the root of which is to be found, $x \div y$ an arbitrary initial value and $x_1 \div y_1$ the succeeding corrected value.

By dividing the right side of (1) by y^n and substituting a for $x \div y$, and a_1 for $x_1 \div y_1$ we obtain another form of the same algorithm:

$$a_1 = \frac{(n-1)a^n + z + naz}{na^{n-1} + a^n + (n-1)z} \tag{2.}$$

In regard to this algorithm the following observation may be allowed. There are the two following algorithms of a more simple form, i. e.

$$a_1 = \frac{(n-1)a^n + z}{na^{n-1}} \quad (3,) \text{ and } \quad a_1 = \frac{naz}{a^n + (n-1)z} \quad (4.)$$

If a be again an arbitrary approximate value for the n^{th} root of z, then the corrected value a_1 in (3) always furnishes a value greater than the n^{th} root of z; and the value in (4) always is smaller than the n^{th} root of z. The acceleration of both methods is of the same (quadratic) order. Now by adding both numerators and dividing by the sum of both denominators we obtain a mean value between the two in (3) and (4); that is, another approximate value for the n^{th} root of z of at least the same degree of approximation· But this result exactly coincides with the algorithm under (2), derived from geometrical constructions.

SOLUTION OF A PROBLEM IN SURVEYING.

BY T. J. LOWRY, U. S. C. S. SAN FRANCISCO, CAL.

Problem:—Four points in the same plane being given in position to determine the position of any other two intervisible points (or places of observation) in reference to these points, having from each place of observation the angles included between the other place of observation and each of two of the known points, and with the known points so situated that the two which are visible from the first point of observation are not visible from the second, and *vice versa*.